(* some simplification constants *)

In[1]:= $T = t_0 * t_1 * (t_0 + t_1)$

Out[1]= $t_0\, t_1\, (t_0 + t_1)$

In[64]:= $X = \Delta x \hat{} 2$

Out[64]= Δx^2

In[69]:= $\alpha = v * \Delta t \,/\, \Delta x$

Out[69]= $\frac{v\, \Delta t}{\Delta x}$

In[68]:= $\beta = b * \Delta t \,/\, 2$

Out[68]= $\frac{b\, \Delta t}{2}$

(* coefficients for 1st-order *)

In[2]:= $a' = -t_1 \hat{} 2 \,/\, T$

Out[2]= $-\frac{t_1}{t_0\, (t_0 + t_1)}$

In[3]:= $b' = (t_1 \hat{} 2 - t_0 \hat{} 2) \,/\, T$

Out[3]= $\frac{-t_0^2 + t_1^2}{t_0\, t_1\, (t_0 + t_1)}$

In[4]:= $c' = t_0 \hat{} 2 \,/\, T$

Out[4]= $\frac{t_0}{t_1\, (t_0 + t_1)}$

(* coefficients for 2nd-order *)

In[39]:= $a'' = 2 * t_1 \,/\, T$

Out[39]= $\frac{2}{t_0\, (t_0 + t_1)}$

In[40]:= $b'' = 2 * (-t_0 - t_1) \,/\, T$

Out[40]= $\frac{2\, (-t_0 - t_1)}{t_0\, t_1\, (t_0 + t_1)}$

In[41]:= $c'' = 2 * t_0 \,/\, T$

Out[41]= $\frac{2}{t_1\, (t_0 + t_1)}$

(* checking 1st-order derivative coefficients *)

In[11]:= Simplify[a' + b' + c']

Out[11]= 0

In[12]:= Simplify[$c' * t_1 - a' * t_0$]

Out[12]= 1

In[45]:=
```
Simplify[a' * t0^2 / 2 + c' * t1^2 / 2]
```

Out[45]= 0

```
(* checking 2nd-order derivative coefficients *)
```

In[42]:=
```
Simplify[a'' + b'' + c'']
```

Out[42]= 0

In[43]:=
```
Simplify[c'' * t1 - a'' * t0]
```

Out[43]= 0

In[46]:=
```
Simplify[a'' * t0^2 / 2 + c'' * t1^2 / 2]
```

Out[46]= 1

```
(* checking that our formulas simplify in the t0=
 t1=h case down to 1st and 2nd order central difference formulas *)
```

In[51]:=
```
Simplify[(a' * f[t - t0] + b' * f[t] + c' * f[t + t1]) /. {t0 → h, t1 → h}]
```

Out[51]= $$\frac{-f[-h+t]+f[h+t]}{2\,h}$$

In[52]:=
```
Simplify[(a'' * f[t - t0] + b'' * f[t] + c'' * f[t + t1]) /. {t0 → h, t1 → h}]
```

Out[52]= $$\frac{-2\,f[t]+f[-h+t]+f[h+t]}{h^2}$$

In[38]:=
```
Series[g[h], {h, x, 2}]
```

Out[38]= $$g[x]+g'[x]\,(h-x)+\frac{1}{2}\,g''[x]\,(h-x)^2+O[h-x]^3$$

```
(* discretized wave equation *)
```

In[70]:=
```
waveEq = a'' * y[i, t - t0] + b'' * y[i, t] + c'' * y[i, t + t1] ==
   v^2 * (y[i - 1, t] - 2 * y[i, t] + y[i + 1, t]) / Δx^2 -
    b * (a' * y[i, t - t0] + b' * y[i, t] + c' * y[i, t + t1])
```

Out[70]= $$\frac{2\,(-t_0-t_1)\,y[i,t]}{t_0\,t_1\,(t_0+t_1)}+\frac{2\,y[i,t-t_0]}{t_0\,(t_0+t_1)}+\frac{2\,y[i,t+t_1]}{t_1\,(t_0+t_1)}==$$
$$-b\left(\frac{\left(-t_0^2+t_1^2\right)y[i,t]}{t_0\,t_1\,(t_0+t_1)}-\frac{t_1\,y[i,t-t_0]}{t_0\,(t_0+t_1)}+\frac{t_0\,y[i,t+t_1]}{t_1\,(t_0+t_1)}\right)+$$
$$\frac{v^2\,(y[-1+i,t]-2\,y[i,t]+y[1+i,t])}{\triangle x^2}$$

In[65]:=
```
handSoln =
  (T / (2 * t0 + b * t0^2)) * (((2 * t0 + 2 * t1 - b * (t1^2 - t0^2)) / T - 2 * v^2 / X) * y[i, t] +
     ((-2 t1 + b * t1^2) / T) * y[i, t - t0] + (v^2 / X) * (y[i - 1, t] + y[i + 1, t]))
```

Out[65]= $$\frac{t_0\,t_1\,(t_0+t_1)\left(\left(-\frac{2\,v^2}{\triangle x^2}+\frac{2\,t_0+2\,t_1-b\left(-t_0^2+t_1^2\right)}{t_0\,t_1\,(t_0+t_1)}\right)y[i,t]+\frac{\left(-2\,t_1+b\,t_1^2\right)y[i,t-t_0]}{t_0\,t_1\,(t_0+t_1)}+\frac{v^2\,(y[-1+i,t]+y[1+i,t])}{\triangle x^2}\right)}{2\,t_0+b\,t_0^2}$$

```
(* verify by-hand results *)
```

In[67]:=
```
FullSimplify[Solve[waveEq, y[i, t + t1]][[1, 1, 2]] == handSoln]
```

Out[67]= True

```
(* check with the t0=
 t1=Δt case to verify that our result is identical to Mike Dubson'
    s previously derived fixed-timestamp formula *)
```

In[80]:=
```
dubsonEq =
 (1 / (β + 1)) * ((β - 1) * y[i, t - Δt] + 2 * (1 - α^2) * y[i, t] + α^2 * (y[i + 1, t] + y[i - 1, t]))
```

Out[80]=

$$\frac{2\left(1-\frac{v^2 \Delta t^2}{\Delta x^2}\right) y[i,t]+\left(-1+\frac{b\,\Delta t}{2}\right) y[i,t-\Delta t]+\frac{v^2 \Delta t^2 (y[-1+i,t]+y[1+i,t])}{\Delta x^2}}{1+\frac{b\,\Delta t}{2}}$$

In[81]:=
```
fixedTimeHandSoln = handSoln /. {t0 → Δt, t1 → Δt}
```

Out[81]=

$$\frac{2\,\Delta t^3\left(\left(\frac{2}{\Delta t^2}-\frac{2 v^2}{\Delta x^2}\right) y[i,t]+\frac{\left(-2\,\Delta t+b\,\Delta t^2\right) y[i,t-\Delta t]}{2\,\Delta t^3}+\frac{v^2 (y[-1+i,t]+y[1+i,t])}{\Delta x^2}\right)}{2\,\Delta t+b\,\Delta t^2}$$

In[82]:=
```
FullSimplify[fixedTimeHandSoln == dubsonEq]
```

Out[82]= True

```
(* check for simpler formulas for our hand-derived formula (result: not really) *)
```

In[88]:=
```
Collect[FullSimplify[handSoln],
  {y[i, t], y[i, t - t0], y[i, t + t1], y[i + 1, t], y[i - 1, t]}] // DisplayForm
```

Out[88]//DisplayForm=

$$\frac{\left(v^2 t_0^2 t_1+v^2 t_0 t_1^2\right) y[-1+i,t]}{\Delta x^2 t_0 (2+b t_0)}+$$

$$\frac{\left(2\,\Delta x^2 t_0+b\,\Delta x^2 t_0^2-2 v^2 t_0^2 t_1-2 v^2 t_0 t_1^2-\Delta x^2 t_1 (-2+b t_1)\right) y[i,t]}{\Delta x^2 t_0 (2+b t_0)}+$$

$$\frac{t_1 (-2+b t_1) y[i,t-t_0]}{t_0 (2+b t_0)}+\frac{\left(v^2 t_0^2 t_1+v^2 t_0 t_1^2\right) y[1+i,t]}{\Delta x^2 t_0 (2+b t_0)}$$